

How to Measure the Fractal Geometry of the Relativistic Fermion Propagator

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April 1995

Université Laval preprint: LAVAL-PHY-3/95

Abstract

We study the geometry of propagation of relativistic fermions. We propose how to measure its quantum mechanical length. Numerical lattice results for the free propagator of Dirac-Wilson fermions yield Hausdorff dimension $d_H = 2$ for the unit-matrix component and $d_H = 1$ for any γ -matrix component. A possible generalization when matter interacts with radiation is discussed.

PACS index: 03.65.-w, 05.30.-d

1. Introduction

What do we know about the geometry of propagation of massive particles in non-relativistic quantum mechanics? According to Feynman and Hibbs [1] quantum mechanical paths are zig-zag lines, which are no-where differentiable, exhibiting self-similarity when viewed at different length scales. In terms of modern language this is a fractal curve [2]. Abbot and Wise [3] have shown for free motion that quantum mechanical paths are fractal curves of Hausdorff dimension $d_H = 2$. Actually, in quantum mechanics the concept of paths is not well defined. If, however, one goes over from real to imaginary time the path integral becomes mathematically well defined (Wiener measure). In imaginary time (Euclidean) quantum mechanics the free motion resembles the Brownian motion of a classical particle [4]. Then one has a stochastic interpretation as classical Brownian motion and paths can be considered as random variables of a Gaussian process [5]. The typical path of a classical particle carrying out a Brownian motion is a fractal curve with $d_H = 2$. According to Itzykson and Drouffe [6] when following a Brownian curve on a lattice from \vec{x}_0, t_0 to \vec{x}_1, t_1 the typical distance behaves as

$$|\vec{x}_1 - \vec{x}_0| \sim (t_1 - t_0)^\nu, \quad \nu = 1/2, \quad (1)$$

and ν is called a critical exponent. (by analogy with the power laws in the theory of critical phenomena). Considering the typical distance as a function of the elementary length Δx then defines the Hausdorff dimension (see Eq.(2) below). It is important to note that $d_H = 2$ for free motion in quantum mechanics does not change when going over from real time to imaginary time.

For the case of a massive interacting quantum mechanical particle Campesino-Romeo et al. [7] have shown analytically $d_H = 2$ for paths of the harmonic oscillator. Other potentials have been investigated by numerical simulations on the lattice [8]. It turned out that $d_H = 2$ for local potentials (harmonic oscillator, Coulomb potential). However, $d_H \neq 2$ was found for velocity-dependent interactions. Such velocity-dependent actions are supposed to play a rôle in condensed matter (dispersion relation of a massive particle travelling in a medium) or in Brueckner's theory of nuclear matter [9].

The Hausdorff dimension of field configurations has recently become a subject of interest in quantum gravity (in particular in 2-D), where the rôle of fields is played by the geometry of space-time [10] - [17]. The Hausdorff dimension can, e.g., be defined as a power law relation between two dimensionful observables at a critical point [10]. E.g., for quantum gravity in D=2, one has $d_H = 4$ [16]. It is interesting to establish relations between the Hausdorff dimension and critical exponents. Such a relation is known to exist between d_H and the critical exponent ν , which is the analogue of a critical exponent of the spin-spin correlation function in statistical mechanics. In quantum gravity [16] ν is defined as the critical exponent of a mass, $m(\Delta\mu) \sim (\Delta\mu)^\nu$, when the cosmological constant tends to its critical value $\Delta\mu \rightarrow 0$, where the mass $m(\Delta\mu)$ characterizes the fall-off behavior of a two-point function $G_\mu(r) \sim \exp[-m(\Delta\mu)r]$. Then holds the following relation (scaling law) $d_H = 1/\nu$, relating the Hausdorff dimension to the critical exponent ν [16]. Thus also d_H plays the rôle of a critical exponent.

Critical exponents of a theory determine its universality class and thus classify the theory. The above example of quantum gravity leads us to the question: Can the Hausdorff dimension play a similar rôle for models like, e.g., theory of matter interacting with radiation, in condensed matter, in the theory of medium energy nuclear physics (Dirac phenomenology), or in high energy for the theory of quarks and gluons (plasma)? A related point of view is the following: In quantum mechanics the Hausdorff dimension of typical paths may differ from the standard value $d_H = 2$ in the presence of velocity-dependent potentials [8], also if one considers quantum mechanics in a background medium corresponding to curved space-time. Such a situation occurs, e.g., when considering the relativistic propagation of a particle in nuclear matter (e.g., a neutron star), or when considering a relativistic particle impinging on the surface and propagating a short distance in ordinary matter.

With those questions in mind we want to start out here by asking the much simpler question: What happens to the geometry of propagation of a massive particle in relativistic quantum mechanics? We want to study the question by considering the most simple massive relativistic particle occurring in nature: the non-interacting Dirac fermion. The purpose of our work is firstly to compute d_H in general and secondly to suggest a definition based on

the propagator which allows for a generalization to interacting fermions.

2. How to measure the geometry of paths in non-relativistic quantum mechanics?

The Hausdorff dimension of a fractal curve is defined as follows: Suppose one has an elementary length scale (resolution) Δx to cover the curve. Experimentally, Δx corresponds to the resolution of an experimental apparatus (yardstick, wavelength of light in microscope). Measuring the length L of a curve in terms of an elementary length Δx , the property of being fractal is captured in the Hausdorff dimension d_H , defined by

$$L_0 \sim_{\epsilon \rightarrow 0} L \epsilon^{d_H-1}, \epsilon = \Delta x / L_1 \quad (2)$$

where L_1 is a fixed length. The Hausdorff dimension d_H is a number chosen such that L_0 becomes independent of ϵ in the limit $\epsilon \rightarrow 0$. Hausdorff has given a precise definition of "resolution" by covering the curve with $L/\Delta x$ spheres of diameter Δx .

The definition given by Eq.(2) has been applied in Ref.[3, 8] to characterize the typical path of a quantum mechanical trajectory and in this sense the Hausdorff dimension has been computed. But in order to be precise one has to say what is meant by length L and resolution Δx for quantum trajectories. Suppose we consider the amplitude for propagation of a particle, being at $t = 0$ in a state characterized by position x_{in} and at $t = T$ in a state characterized by position x_{fi} . We discretize time $t_0 = 0 < t_1 < \dots < t_N = T$, with $\Delta t = \delta$ and $T = N\Delta t$. Then the amplitude is given by a path integral, which after discretization of time reads

$$Z(\delta) = \int dx_1 \cdots dx_N \exp[iS[x_k, \delta]/\hbar] \Big|_{x_0=x_{in}, x_N=x_{fi}}. \quad (3)$$

We have denoted $x_k = x(t_k)$. This amplitude is an approximation of the continuum (exact) amplitude, obtained by taking the limit $\Delta t \rightarrow 0$, i.e., $N \rightarrow \infty$ in an appropriate way. A suitable observable to study the geometry is the propagator length defined by [8]

$$\begin{aligned} \langle L(\delta) \rangle &= \left\langle \sum_{k=0}^{N-1} |x_{k+1} - x_k| \right\rangle \Big|_{x_0=x_{in}, x_N=x_{fi}} \\ &= \frac{1}{Z} \int dx_1 \cdots dx_N \sum_{k=0}^{N-1} |x_{k+1} - x_k| \exp[iS[x_k, \delta]/\hbar] \Big|_{x_0=x_{in}, x_N=x_{fi}}. \end{aligned} \quad (4)$$

In quantum mechanics, the resolution of length is given by the dynamics. It is natural to define it as average length increment corresponding to a time increment δ [3],

$$\langle \Delta x \rangle = \frac{1}{N} \left\langle \sum_{k=0}^{N-1} |x_{k+1} - x_k| \right\rangle_{x_0=x_{in}, x_N=x_{fi}}. \quad (5)$$

As fixed length L_1 one can choose, e.g., the length of the classical trajectory from $x_{in}, t = 0$ to $x_{fi}, t = T$ given by the classical continuum action. If one is interested only in d_H , i.e., the exponent of the power law Eq.(2), then any fixed length L_1 can be chosen, in particular the length of the straight line between x_{in} and x_{fi} . In the continuum limit of quantum mechanics one takes the limit $\Delta t = \delta \rightarrow 0$, which implies $\langle \Delta x \rangle \sim \epsilon \rightarrow 0$. The length $\langle L \rangle$ has been computed by numerical simulations on the lattice using imaginary time (Euclidean) quantum mechanics and d_H has been extracted [8].

3. Definition of propagation length in relativistic quantum mechanics

When trying to generalize the above considerations to relativistic quantum mechanics, one is faced with the following problem: (a) In quantum mechanics, position of a particle is an observable and can be measured. In quantum field theory position is not an observable. (b) The particle number is conserved in quantum mechanics. In quantum field theory particles can be created and annihilated. (c) In quantum mechanics particles propagate only forward in time. In quantum field theory particles propagate forward and backward (anti-particles) in time. Thus in a relativistic theory one must look at space and time dependence, corresponding to a causal propagation of a massive particle. From the mathematical point of view as we have pointed out above, the path integral in imaginary time quantum mechanics is well defined in terms of a stochastic process. This can be generalized to Euclidean path integrals of bosonic (polynomial) field theory which are mathematically well defined, allowing an interpretation as stochastic process [18, 5]. The measure gives the dominant contributions of nowhere differentiable curves [18]. However, for fermion field theory, there is not yet a strict mathematical formulation in terms of a stochastic process, although Osterwalder and Schrader [19] have established a Feynman-Kac formula for fermion fields.

From those remarks it is evident that the definition of propagator length given by Eq.(4) for non-relativistic quantum mechanics can not be simply taken over to relativistic quantum mechanics. The definition (4) is based on a discretization of time (taking $\Delta t = \delta$ finite and letting $\delta \rightarrow 0$ in the end when extracting the critical exponent). In relativistic quantum mechanics we discretize time and space, i.e., we work on a lattice with some finite lattice spacing a (letting $a \rightarrow 0$ in the end). Let us introduce a new definition for the length of propagation for relativistic quantum mechanics, by considering, in particular, the Dirac fermion action. It is well known that the corresponding lattice action is plagued by the so-called fermion species doubling problem: 16 copies of fermions with the same mass occur (as poles of the fermion propagator). A way out commonly used is the Wilson action, which lifts the species doubling by adding another term. The (Euclidean) Wilson-Dirac action on the lattice reads [20]

$$S[\psi, \bar{\psi}] = \sum_{m,n} \bar{\psi}_m K_{m,n} \psi_n. \quad (6)$$

Here m, n are indices (tupel) which denote lattice sites (e.g., $x_n = na$, $n = 0, \pm 1, \dots$, in $D = 1$). The fields have been rescaled $a^{3/2}(am + 4r)^{1/2}\psi \rightarrow \psi$, $\kappa = 1/(2ma + 8r)$ is the hopping parameter and r is the Wilson parameter (usually chosen to be $r = 1$). The matrix K is expressed in terms of the hopping matrix M

$$\begin{aligned} K_{m,n} &= \delta_{m,n} - \kappa M_{m,n}, \\ M_{m,n} &= \sum_{\mu=1}^4 (r + \gamma_\mu) \delta_{m+\hat{\mu},n} + (r - \gamma_\mu) \delta_{m-\hat{\mu},n}. \end{aligned} \quad (7)$$

Here $m + \hat{\mu}$ denotes the lattice site next to site m in the positive μ -direction. The fermion propagator is given by

$$\langle \psi_n \bar{\psi}_m \rangle = (K^{-1})_{n,m}. \quad (8)$$

The matrix M , coming from the kinetic term of the fermion action and from the Wilson term, allows the fermion to hop from one lattice site to the next neighbour lattice site. We suggest to define the length of the fermion propagator by counting in a non-perturbative way the number of hoppings. In particular we suggest the following definition

$$\langle L_{m,n} \rangle = \frac{\partial \log \langle \psi_n \bar{\psi}_m \rangle}{\partial \log \kappa}. \quad (9)$$

The indices m, n denote the lattice sites, where the fermion is created and annihilated, respectively. By expanding the right-hand side of Eq.(9) as a power series in the hopping parameter, one finds that the p -th power of the hopping matrix M , which allows the fermion to hop a distance pa , gets multiplied with a factor p . Thus $\langle L_{m,n} \rangle$ can be interpreted as a (non-perturbative) counter of how many times a fermion hops between sites m and n .

The classical length is $L_{class} = |x_m - x_n|$. One has to compute numerically on the lattice $\langle L \rangle$ as a function of a/L_{class} . The goal is to extract a critical exponent γ ,

$$\frac{\langle L \rangle}{L_{class}} \sim_{a/L_{class} \rightarrow 0} (a/L_{class})^{-\gamma}. \quad (10)$$

By Eq.(2), γ is related to d_H via $d_H = 1 + \gamma$. Because action (6) is parametrized in terms of the hopping parameter κ , it is natural to consider another critical exponent α defined by

$$\frac{\langle L \rangle}{L_{class}} \sim_{\kappa \rightarrow \kappa_{crit}} \left(\frac{\kappa_{crit} - \kappa}{\kappa_{crit}} \right)^{-\alpha}. \quad (11)$$

The critical exponent γ is defined in the continuum limit $a \rightarrow 0$. When a goes to zero, the dimensionless lattice mass ma goes to zero, and κ goes to its critical value $\kappa_{crit} = 1/8r$. For a free Euclidean fermion theory, which we investigate numerically, both exponents coincide, $\alpha = \gamma$.

4. Numerical results

Before measuring the length of the fermion propagator, it is useful to see if the definition of the propagator length makes sense. In order to have a meaningful length definition, one would expect $\langle L \rangle$ to obey a power law (11) when approaching κ_{crit} for fixed L_{class} . In order to get a first idea on the behavior of $\langle L \rangle$ we have considered a drastically simplified hopping matrix $M_{i,j} = \delta_{i,j+1} + \delta_{i+1,j}$. We have dropped any dependence from γ -matrices and study the length as a function of κ . We have done numerical calculations for $D = 1$ with $\kappa_{crit} = 1/2$ on lattices up to $N = 120$ and for $D = 2$ with $\kappa_{crit} = 1/4$ on lattices up to $N = 50$. The numerical results confirm the expected scaling behavior of Eq.(11). the results yield $\alpha = 0.49$ for both $D = 1$ and $D = 2$.

Now we turn to the fermion propagator. For the free fermion case, $\kappa_{crit} = 1/8r$ in $D = 4$. The Euclidean free fermion propagator for Wilson fermions is given in momentum space by [20]

$$\tilde{\Delta}_k = \left(1 - 2r\kappa \sum_{\mu=1}^4 \cos k_\mu + 2\kappa \sum_{\mu=1}^4 i\gamma_\mu \sin k_\mu \right)^{-1}. \quad (12)$$

This is related to the space-time propagator $\Delta_{x,y} \equiv \langle \psi_x \bar{\psi}_y \rangle$ by Fourier transformation

$$\Delta_{x,y} = \frac{1}{V} \sum_k e^{ik \cdot (x-y)} \tilde{\Delta}_k, \quad (13)$$

where $V = N_1 N_2 N_3 N_4$ is the lattice volume. Note that $\tilde{\Delta}_k$ has a pole at $k = 0$, $\kappa_{crit} = 1/2Dr$ in D space-time dimensions. We choose periodic or anti-periodic boundary conditions. They correspond to the following choice of lattice momenta k_μ (see Ref.[20]), $k_\mu = 2\pi n_\mu / N_\mu$ corresponds to periodic boundary conditions and $k_\mu = 2\pi(n_\mu + 1/2)/N_\mu$ corresponds to anti-periodic boundary conditions. In both cases, $n_\mu \in 0, 1, \dots, N_\mu - 1$. Thus in the anti-periodic case $k_\mu^{min} = \pi/N_\mu$ is the minimal value of k_μ .

We consider two components of the propagator: the unit-matrix component is given by $\frac{1}{4}Tr[\Delta_{x,y}]$ and the γ_μ -component by $\frac{1}{4}Tr[\gamma_\mu \Delta_{x,y}]$. Let us consider the cases of $D = 1$, $D = 2$ and $D = 4$ space-time dimensions. For $D = 1$, one can compute the large lattice limit ($V \rightarrow \infty$, not the continuum limit) analytically and obtains

$$\begin{aligned} \frac{1}{4}Tr[\Delta_{x-y=n}] &= 2^{n-1} \kappa^n, \quad 1 \leq n, \quad \kappa \leq \kappa_{crit}, \\ < L > &= L_{class}. \end{aligned} \quad (14)$$

However, we are interested in the continuum limit $a \rightarrow 0$, which corresponds to $\kappa \rightarrow \kappa_{crit}$, with $\kappa_{crit} = 1/2r$ in $D = 1$. The Wilson parameter is $r = 1$. The numerical results for the γ_1 -component and the unit-component are shown in Fig.[1a,b]. For the γ_1 -component, we have chosen anti-periodic boundary conditions. We have varied $N \equiv N_1 = 4, 8, 16, \dots, 1024$. In order to approach κ_{crit} we have varied $\kappa = 1/[2r \cos(k) + 2 \sin(k)]$, with $k = k^{min} = \pi/N$. We have chosen as classical length $L_{class} = |x - y| = 1$. We have evaluated the space-time propagator (13) by doing the Fourier transformation of $\tilde{\Delta}_k$ and of $\frac{d}{d\kappa} \tilde{\Delta}_k$ numerically. From that we have evaluated the length $< L >$ via Eq.(9) and

hence the exponent α and d_h via Eq.(11). The result shows a power law behavior (11) with $\alpha = -0.0016$, which corresponds to $d_H = 0.9984$, by Eq.(2).

The behavior of the unit-component, with periodic boundary conditions, is different. We have varied $N = 20, 40, \dots, 100$. Because for periodic boundary conditions $k^{min} = 0$, we have approached $\kappa \rightarrow \kappa_{crit}$ by choosing $\kappa = 0.45, 0.475, 0.4875, \dots$ (decreasing $|\kappa_{crit} - \kappa|$ by a factor 2 in each step.). Now we have considered the classical length $L_{class} = N/2$. The computation of $\langle L \rangle$ and d_H is as above. One observes (Fig.[1b]) a power law with the critical exponent varying between $\alpha = 1.000$ and $\alpha = 0.9983$, and the corresponding fractal dimension varying between $d_H = 2.000$ and $d_H = 1.9983$. Generally, one observes that the larger the size of the lattice, the closer one has to be at κ_{crit} before the scaling behavior (11) is seen. How can a curve in topological dimension $D = 1$ show a fractal dimension larger than 1? The Hausdorff dimension measures the hopping of the fermion forward and backward on a line ($D = 1$), which can be fractal.

Similar results are obtained in $D = 2$ space-time dimensions, shown in Fig.[2a,b]. Now $\kappa_{crit} = 1/4r$. Firstly, we have considered the γ_2 -component. Because we have a regular, symmetric lattice, the k -dependence is the same for all γ_μ -components. Thus we can interpret the γ_1 -component as space component and the γ_2 -component as time-component. We have chosen boundary conditions periodic in space and anti-periodic in time. We have varied $N_1 = N_{space} = 4, 8, 16$ independently from $N_2 = N_{time} = 4, 8, 16, \dots, 1024$. In order to approach κ_{crit} we have varied $\kappa = 1/[2r(1+\cos(k))+2\sin(k)]$, where $k = k_2^{min} = \pi/N_{time}$. As classical length we have chosen $L_{class} = N_{time}/2$. We have obtained $\alpha = 0.0022$ and $d_H = 1.0022$.

For the unit-component, we have chosen periodic boundary conditions in space and time. We have varied $N = N_1 = N_2 = 10, 20, \dots, 50$. In order to approach κ_{crit} we have chosen $\kappa = 0.225, 0.2375, \dots$ (decreasing $|\kappa_{crit} - \kappa|$ by a factor 2 in each step). As classical length we have chosen $L_{class} = N/2$. We find $d_H = 1.999$ to $d_h = 1.994$ for lattices varying between $N = 10, \dots, 50$.

Finally, we present the results for $D = 4$ in Fig.[3a,b]. Now $\kappa_{crit} = 1/8r$. For the γ_4 -component, we have chosen boundary conditions periodic in space and anti-periodic in time.

We have varied $N_1 = N_2 = N_3 = N_{space} = 4, 8, 16$ and $N_4 = N_{time} = 4, 8, \dots, 256$. In order to approach κ_{crit} we have varied $\kappa = 1/[2r(3+\cos(k))+2\sin(k)]$, where $k = k_4^{min} = \pi/N_{time}$. As classical length we have chosen $L_{class} = N_{time}/2$. As results we obtain $\alpha = 0.0086$ and $d_H = 1.008$.

For the unit-component, we have chosen periodic boundary conditions in space and time. We have varied $N = N_1 = \dots = N_4 = 4, 8, 12, \dots, 28$. In order to approach κ_{crit} we have chosen $\kappa = 0.100, 0.110, \dots$ (decreasing $|\kappa_{crit} - \kappa|$ by a factor 2 in each step). As classical length we have chosen $L_{class} = N/2$. We find $d_H = 2.10$ to $d_h = 2.05$ for lattices varying between $N = 4, \dots, 28$.

One obtains the following picture: The γ_μ -component of the propagator shows results compatible with $d_H = 1$, i.e., no fractal behavior, in $D = 1, 2, 4$ and different combinations of periodic/anti-periodic boundary conditions. However, the unit-component shows results compatible with $d_H = 2$ for $D = 1, 2, 4$, i.e., the same fractal behavior as in non-relativistic quantum mechanics. The numerical results, that is $d_H = 1$ for the unit-component and $d_H = 2$ for γ_μ -component are independent of these boundary conditions. For larger lattices, scaling sets in later (closer to κ_{crit}). Numerical errors increase when approaching the singularity $\kappa \rightarrow \kappa_{crit}$. Also, numerical errors increase with the size of the lattice. Nevertheless, one observes for $D = 4$ that when increasing the lattice size, the Hausdorff dimension moves closer to the value 2.

5. Discussion

(a) Why differ the results of d_H for different components? Let us consider periodic boundary conditions and compare the unit-component with the γ_1 -component of the propagator. Then the propagator, given in momentum space by Eq.(12), projects under the Fourier transformation (13) onto the $\cos(k \cdot (x - y))$ part for the unit-component, but onto the $\sin(k \cdot (x - y))$ part for the γ_1 -component. In other words, the unit- and the γ_1 -component have a different pole structure when $k \rightarrow 0$. Let us consider the continuum limit, $a \rightarrow 0$ and $\epsilon = \kappa_{crit} - \kappa \rightarrow 0$, but keep the lattice volume V fixed. Also we keep $L_{class} = |x - y| = \text{const.}$

and consider in $D = 1$ the unit-component $\frac{1}{4}Tr[\Delta_{x,y}]$, given by Eqs.(12,13), as a function of ϵ . For small enough ϵ , the dominant contributions come from lattice momenta k_i with $k_i \ll \epsilon$. Taking in Eq.(13) only those lattice momenta into account, one finds $\frac{1}{4}Tr[\Delta_{x,y}] \sim 1/\epsilon$. Consequently, $\kappa \frac{d}{d\kappa} \frac{1}{4}Tr[\Delta_{x,y}] \sim 1/\epsilon^2$. Thus $\langle L \rangle \sim 1/\epsilon$, which by Eq.(11) implies $d_H = 2$. Doing the analogous calculation for the γ_1 -component yields $\frac{1}{4}Tr[\gamma_1 \Delta_{x,y}] \sim \text{const.}$ and $\kappa \frac{d}{d\kappa} \frac{1}{4}Tr[\gamma_1 \Delta_{x,y}] \sim \text{const.}$. Thus $\langle L \rangle \sim \text{const.}$ and hence $d_H = 1$. This is in agreement with our numerical results.

(b) The results for the free fermion propagator on the lattice can be compared with Feynman's analytical expression for the fermion propagator in the asymptotic regime $x^2 \ll t^2$ and $x^2 \gg t^2$ [21]. Feynman expresses the propagator kernel $K_+(2,1) = i(i\gamma_\mu \partial^\mu + m)I_+(t, \vec{x})$, and gives for the function I_+ the asymptotic expression

$$I_+(t, \vec{x}) \rightarrow \exp\{-im[t - (x^2/2t)]\}, \quad x^2 \ll t^2, \quad (15)$$

It can be seen that the propagation kernel is essentially the same as for a free particle in non-relativistic quantum mechanics, where the Hausdorff dimension is $d_H = 2$. This is in accord with our result $d_H = 2$ for the unit-component of the fermion propagator, which dominates the non-relativistic regime.

(c) The definition of length (9) for the action (6) can be generalized to the case when matter interacts with radiation, i.e., *QED*. Then the fermion-photon interaction has the same structure as in Eq.(6), but the hopping matrix $M[U]$ depends now on the gauge field via the link variables $U_\mu(n)$ (for details see [20]), and the matrix element $(K^{-1})_{m,n}$ occurring in the fermion propagator, Eq.(8), must be replaced by a quantum expectation value $\langle (K[U]^{-1})_{m,n} \rangle$ which means doing a path integral over the gauge field. However, one has to fix the gauge (see [22]).

In summary, we have suggested a definition of length for the propagation of relativistic fermions. It shows scaling behavior when approaching the continuum limit and yields the critical exponents $d_H = 2$ (unit-component) and $d_H = 1$ (γ_μ -component). This is consistent with the analytical behavior of the Fermion propagator. Our length definition can be directly generalized to interacting theories, e.g., matter with radiation.

Acknowledgement

The author is grateful to J. Polonyi for very stimulating discussions on the propagation length. The author gratefully acknowledges support from NSERC Canada.

References

- [1] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw Hill, New York (1965).
- [2] B.B. Mandelbrot, The Fractal Geometry of Nature, Freeman, New York (1983).
- [3] L.F. Abbot and M.B. Wise, Am. J. Phys. 49(1981)37.
- [4] E. Nelson, Phys. Rev. 150(1966)1079.
- [5] G. Roepstorff, Path Integral Approach to Quantum Mechanics, Springer, New York (1994).
- [6] C. Itzykson and J.B. Drouffe, Statistical Field Theory, Cambridge Univ. Press, Cambridge (1989), Vol.I.
- [7] E. Campesino-Romeo, J.C.D'Olivio and M. Sokolovsky, Phys. Lett. A89(1982)321.
- [8] H. Kröger, S. Lantagne, K.J.M. Moriarty, and B. Plache, Phys. Lett. A199(1995)299.
- [9] K.A. Brueckner, Phys. Rev. 97(1955)1353.
- [10] H. Kawai and M. Ninomiya, Nucl. Phys. B336(1990)115.
- [11] F. David, Nucl. Phys. B368(1992)671.
- [12] N. Kawamoto, V.A. Kazakov, Y. Saeki, Y. Watabiki, Phys. Rev. Lett. 68(1992)2113.
- [13] H. Kawai, N. Kawamoto, T. Mogami, Y. Watabiki, Phys. Lett. B306(1993)19.
- [14] J. Ambjorn , G. Thorleifson, Phys. Lett. B323(1994)7.
- [15] J. Nishimura, M. Oshikawa, Phys. Lett. B338(1994)187.
- [16] J. Ambjorn , Y. Watabiki, Nils Bohr Inst. preprint NBI-HE-95-01.
- [17] S. Catteral, G. Thorleifson, M. Bowick, V. John, Syracuse Univ. preprint (1995) SU-4240-607.

- [18] J. Glimm and A. Jaffe, Quantum Physics, Springer, New York (1981).
- [19] K. Osterwalder, R. Schrader, Phys. Rev. Lett. 29(1972)1423; Helv. Phys. Acta 46(1973)277; Comm. Math. Phys. 31(1973)83.
- [20] I. Montvay and G. Münster, Quantum Fields on a Lattice, Cambridge Univ. Press, Cambridge (1994).
- [21] R.P. Feynman, Quantum Electrodynamics, Benjamin, New York (1961).
- [22] C. Bernard, D. Murphy, A. Soni, K. Yee, Nucl. Phys. B(Proc. Suppl.)17(1990)593.

Figure Caption

Fig.1 $\langle L \rangle / L_{class}$ versus $(\kappa_{crit} - \kappa) / \kappa_{crit}$ for free fermion propagator in D=1, (a) unit component, (b) γ_1 -component of propagator.

Fig.2 Same as Fig.[1] in D=2 dimensions, (a) unit-component, (b) γ_2 -component.

Fig.3 Same as Fig.[1] in D=4 dimensions, (a) unit-component, (b) γ_4 -component.











